THE MAYER-VIETORIS LONG EXACT SEQUENCE

THEOREM

Let X be a space and $A, B \subset X$ such that $\mathring{A} \cup \mathring{B} = X$. Let $\mathcal{U}_0 = \{A, B\}$, and i^A, i^B, j^A, j^B the following inclusions:

$$\begin{tikzcd} & A \arrow[dr, "j^A"] & \\ A \cap B \arrow[ur, "i^A"] \arrow[dr, "i^B"'] & & A \cup B = X \\ & B \arrow[ur, "j^B"'] & \end{tikzcd}$$

Then, the sequence $(*)$

$$0 \to C_n(A \cap B) \xrightarrow{i_C^A \oplus (-i_C^B)} C_n(A) \oplus C_n(B) \xrightarrow{j_C^A + j_C^B} C_n^{\mathcal{U}_0}(X) \to 0$$

is an exact sequence of chain complexes.

In particular, it induces a LES in

homology

$$\cdots \to H_p(A\cap B) \xrightarrow{i_*^A \oplus (-i_*^B)} H_p(A)\oplus H_p(B) \xrightarrow{j_*^A + j_*^B} H_p(X) \to$$

$$\to H_{p-1}(A\cap B) \to \cdots \qquad (**)$$

If $A\cap B \neq \emptyset$ then the same sequence with reduced homologies is also exact.

Proof

Let us first check that $\circledast$ is exact.

$i_C^A \oplus (-i_C^B)$ is injective ✓

$j_C^A + j_C^B$ is surjective ✓

Now let $c_A \oplus c_B \in \ker(j_C^A + j_C^B)$, where $c_A \in S_p(A)$ & $c_B \in S_p(B)$.

$\Rightarrow j_C^A(c_A) + j_C^B(c_B) = c_A + c_B = 0$

$\Rightarrow c_B = -c_A$. this implies that

$c_A, c_B \in S_p(A\cap B)$. $\Rightarrow \exists c \in S_p(A\cap B)$ s.t.

$(i_C^A \oplus (-i_C^B))(c) = c_A \oplus c_B$

Finally, note that $H_p(C_n^{\mathcal{U}}(X)) \cong H_p(X)$ by the map induced by the inclusion $C_n^{\mathcal{U}}(X) \to C_n(X)$. Now the statement folows from the SES $\Rightarrow$ LES.

For relative homology one studies the LES of

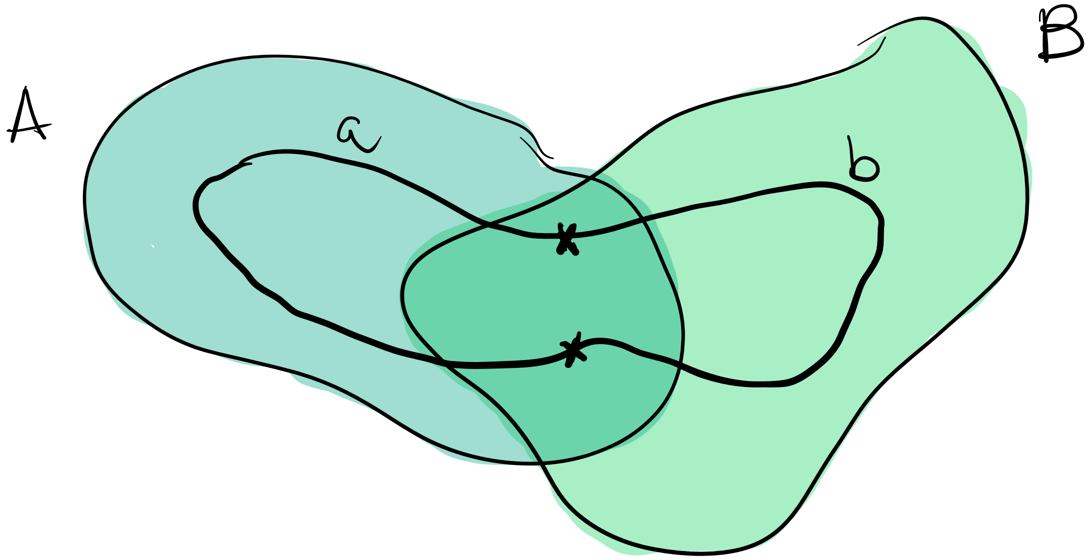
$$\begin{tikzcd}
 & {} \arrow[d, "\partial"] & {} \arrow[d, "\partial"] & {} \arrow[d, "\partial"] & \\
0 \arrow[r] & S_0(A \cap B) \arrow[r] \arrow[d, "\varepsilon"] & S_0(A) \oplus S_0(B) \arrow[r] \arrow[d, "\varepsilon \oplus \varepsilon"] & S_0(A + B) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

∎

We can show how $\partial_*: H_n(X) \to H_{n-1}(A \cap B)$ behaves geometrically. For any $\sigma \in H_n(X)$, select a cycle c s.t.

$$\mu = [c].$$

We can select such a c that $c = a + b$, where $a \in S_p(A)$, $b \in S_p(B)$.



(this we can do since we can use barycentric subdivision to break down c into simplices of as small diameter as desired that still represent the same homology class)

$$\begin{tikzcd}
 & a \oplus b \arrow[r, mapsto] & c = a+b & \\
 & S_p(A) \oplus S_p(B) \arrow[r] \arrow[d] & S_p(A+B) \arrow[r] & 0 \\
S_{p-1}(A\cap B) \arrow[r] & S_{p-1}(A) \oplus S_{p-1}(B) & & \\
\partial a \arrow[r, mapsto] & \partial(a \oplus b) = \partial a \oplus \partial b & &
\end{tikzcd}$$

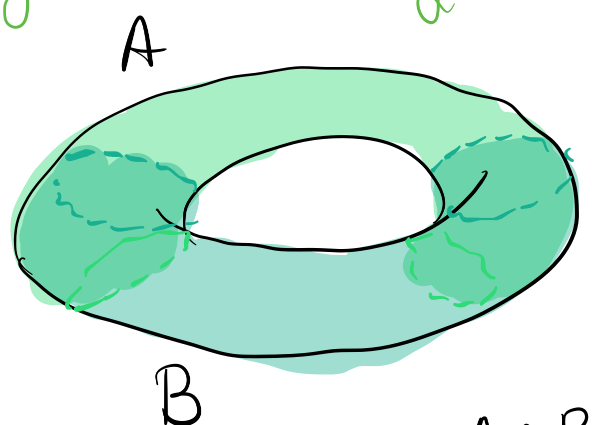
Since c is a cycle

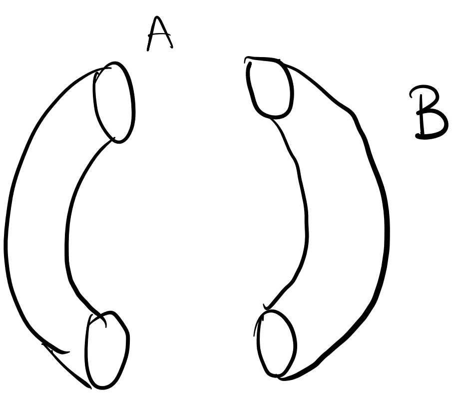
$0 = \partial c = \partial a + \partial b$ and therefore, $-\partial b = \partial a$.

$$\partial_* m = [\partial a] = [-\partial b] \in H_{p-1}(A \cap B).$$

EXAMPLE

Compute homology groups of the torus using the Mayer-Vietoris sequence.





$A \cap B$ homotopy equivalent to $S^1 \sqcup S^1$

A, B homotopy equivalent to S^1